SELMER GROUPS OF ABELIAN VARIETIES IN EXTENSIONS OF FUNCTION FIELDS

AMÍLCAR PACHECO

ABSTRACT. Let k be a field of characteristic q, $\mathcal{C}$ a smooth geometrically connected curve defined over k with function field $K := k(\mathcal{C})$. Let A/K be a non constant abelian variety defined over K of dimension d. We assume that $q = 0$ or $> 2d+1$. Let $p \neq q$ be a prime number and $\mathcal{C}' \to \mathcal{C}$ a finite geometrically GALOIS and étale cover defined over k with function field $K' := k(\mathcal{C}')$. Let (τ', B') be the K'/k-trace of A/K. We give an upper bound for the $\mathbb{Z}_p$-corank of the SELMER group $\mathrm{Sel}_p(A \times_K K')$, defined in terms of the p-descent map. As a consequence, we get an upper bound for the $\mathbb{Z}$-rank of the LANG-NÉRON group $A(K')/\tau' B'(k)$. In the case of a geometric tower of curves whose GALOIS group is isomorphic to $\mathbb{Z}_p$, we give sufficient conditions for the LANG-NÉRON group of A to be uniformly bounded along the tower.

1. INTRODUCTION

Let $\mathcal{C}$ be a smooth geometrically connected (not necessarily complete) curve defined over a field k of characteristic $q \geq 0$. Let $\mathcal{X}$ be a regular compactification of $\mathcal{C}$. Denote by $K := k(\mathcal{X}) = k(\mathcal{C})$ the function field of $\mathcal{X}$ (or equivalently of $\mathcal{C}$) and let g be the genus of $\mathcal{X}$. Let k^s be a separable closure of k, $\mathcal{K} := k^s(\mathcal{X}) = k^s(\mathcal{C})$ and $\mathcal{K}^s$ a separable closure of $\mathcal{K}$.

Let A/K be a non constant abelian variety of dimension d defined over K. A model for A/K over k consists of a smooth geometrically connected projective variety $\mathcal{A}$ defined over k and a proper flat morphism $\phi : \mathcal{A} \to \mathcal{C}$ also defined over k whose generic fiber is A/K. Denote by $\mathcal{U}$ the smooth locus of ϕ and $\mathcal{A}_{\mathcal{U}} := \phi^{-1}(\mathcal{U})$. The morphism ϕ induces an abelian scheme $\phi_{\mathcal{U}} : \mathcal{A}_{\mathcal{U}} \to \mathcal{U}$ having still A/K as its generic fiber.

Let (τ, B) be the K/k-trace of A and $d_0 := \dim(B)$. A theorem of LANG and NÉRON [La83, theorem 2, chapter 6] states the group $A(\mathcal{K})/\tau B(k^s)$ is finitely generated. *A fortiori*, the same holds for $A(K)/\tau B(k)$.

Let $\mathfrak{F}_A$ be the conductor divisor of A/K in $\mathcal{X}$ and f_A its degree. OGG obtained in [Og62, theorem 2] the following geometric upper bound for the rank of $A(\mathcal{K})/\tau B(k^s)$

$$\text{rank}\left(\frac{A(\mathcal{K})}{\tau B(k^s)}\right) \leq 2d(2g-2) + f_A + 4d_0. \tag{1.0.1}$$

In particular, this is also an upper bound for the rank of $A(K)/\tau B(k)$.

Let $\psi : \mathcal{C}' \to \mathcal{C}$ be a finite morphism defined over k which is also a geometrically GALOIS cover. Let $\mathcal{G} := \mathrm{Aut}_{k^s}(\mathcal{C}'/\mathcal{C})$ be its GALOIS group. The absolute GALOIS

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group $G_k := \mathrm{Gal}(k^s/k)$ of k acts on $\mathcal{G}$ and we denote by $\mathfrak{O}_{G_k}(\mathcal{G})$ the set of orbits of this action.

Let $K' := k(\mathcal{C}')$, g' the genus of $\mathcal{C}'$ and (τ', B') the K'/k-trace of A. If k is a number field, $\mathcal{C}$ is complete and ψ is geometrically abelian, we obtained in [Pa05, theorem 1.3] the following upper bound

$$\text{rank}\left(\frac{A(K')}{\tau' B'(k)}\right) \leq \frac{\#\mathfrak{O}_{G_k}(\mathcal{G})}{\#\mathcal{G}}(d(2d+1)(2g'-2)) + \#\mathfrak{O}_{G_k}(\mathcal{G})(2df_A). \tag{1.0.2}$$

This result was proved under the assumptions that the TATE's conjecture for divisors holds for $\mathcal{A}/k$ and that the monodromy representations with respect to the first and second étale cohomology groups of A/K with coefficients in $\mathbb{Q}_\ell$ are irreducible. For an explanation of these two hypotheses see *loc. cit.* sections 2 and 3. In the particular case where A is the Jacobian variety of a curve and ψ is étale, then the upper bound in (1.0.2) is indeed better than that in (1.0.1) (*loc. cit.* paragraph after remark 1.8, in particular formula (1.8)).

In this paper we obtain an improvement the bound (1.0.2) in theorem 3.10 without appeal to the previous hypotheses and of the same order of magnitude as that of [Pa05, (1.8)], where it was only obtained in the case where A was a Jacobian variety. This extends a previous result of ELLENBERG in [El06, theorem 2.8] from elliptic to abelian fibrations. In order to do this we start with descent maps in §2, then in §3 we treat SELMER groups and prove our first result. In §4 we give conditions for the LANG-NÉRON groups of A to be uniformly bounded along a geometric tower of curves over k whose GALOIS group is isomorphic to $\mathbb{Z}_p$. Our theorem 4.5 extends [El06, theorem 4.4]. In the final section (§5) we discuss a sufficient condition to obtain an example (related to Jacobian fibrations) where the hypotheses of theorem 4.5 are met.

2. Descent maps

Let $\mathcal{C}_s := \mathcal{C} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^s)$, $\mathcal{U}_s := \mathcal{U} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^s)$ and η_s the geometric generic point of $\mathcal{U}$. Let $\pi_1(\mathcal{U}_s, \eta_s)$ be the *algebraic fundamental group* of $\mathcal{U}_s$ with respect to η_s. Let $\mathcal{K}^{\mathrm{ur}}$ the maximal GALOIS subextension of $\mathcal{K}^s/\mathcal{K}$ that is unramified over $\mathcal{U}_s$. Then $\pi_1(\mathcal{U}_s, \eta_s) \cong \mathrm{Gal}(\mathcal{K}^{\mathrm{ur}}/\mathcal{K})$. Let $\pi_1^t(\mathcal{U}_s, \eta_s)$ be the *tame algebraic fundamental group* of $\mathcal{U}_s$ with respect to η_s. Let $\mathcal{K}^t$ be the maximal GALOIS subextension of $\mathcal{K}^s/\mathcal{K}$ which is unramified over $\mathcal{U}_s$ and at most tamely ramified over $\mathcal{C}_s \setminus \mathcal{U}_s$. Then $\pi_1^t(\mathcal{U}_s, \eta_s) \cong \mathrm{Gal}(\mathcal{K}^t/\mathcal{K})$.

Let $p \neq q := \mathrm{char}(k)$ a prime number. For each integer $n \geq 1$, let $A[p^n]$ be the subgroup of p^n-torsion points of A and $A[p^\infty] := \bigcup_{n \geq 1} A[p^n] = \varinjlim_n A[p^n]$ the p-divisible subgroup of A. Let $\mathcal{K}(A[p^n])$ be the subfield of $\mathcal{K}^s$ generated over $\mathcal{K}$ by the coordinates of the points in $A[p^n]$. By the NÉRON-OGG-SHAFAREVICH criterion [SeTa68, theorem 1] for every $n \geq 1$ the group $A[p^n]$ is unramified at every $v \in \mathcal{U}_s$, therefore $\mathcal{K}(A[p^n]) \subset \mathcal{K}^{\mathrm{ur}}$.

For each $v \in \mathcal{C}_s$, let $\mathcal{K}_v$ be the completion of $\mathcal{K}$ at v. Let v^s be the unique extension of v to $\mathcal{K}^s$ and $\mathcal{K}^s_{v^s}$ the completion of $\mathcal{K}^s$ at v^s. The inertia group I_v at v equals the decomposition group D_v at v, since k^s is separably closed. Moreover, D_v is isomorphic to $\mathrm{Gal}(\mathcal{K}^s_{v^s}/\mathcal{K}_v)$.

Lemma 2.1. *For every integer $n \geq 1$, we have a short exact sequence of groups*

$$0 \to A[p^n] \to A(\mathcal{K}^{\mathrm{ur}}) \xrightarrow{\times p^n} A(\mathcal{K}^{\mathrm{ur}}) \to 0. \tag{2.1.1}$$

Proof. It suffices to prove the surjectivity of the multiplication by p^n map. Let $x \in A(\mathcal{K}^{\mathrm{ur}})$ and $y \in A(\mathcal{K}^s)$ such that $p^n y = x$. Let $v \in \mathcal{U}_s$ and $\sigma \in I_v$. Then $\sigma(p^n y) = p^n \sigma(y) = \sigma(x) = x = p^n y$, thus $\sigma(y) - y \in A[p^n]$.

Let $\mathcal{L} := \mathcal{K}_v((1/p^n)x)$ be the Galois subextension of $\mathcal{K}^s_{v^s}/\mathcal{K}_v$ generated by all the solutions $y' \in A(\mathcal{K}^s_{v^s})$ of $p^n y' = x$ over $\mathcal{K}_v$. Let w be the extension of v to $\mathcal{L}$ and $\mathcal{O}_w$ its valuation ring.

Let $\mathbf{A}_w$ be the Néron minimal model of A at w and $\tilde{\mathbf{A}}_w$ its special fiber defined over k^s. By the properties of the Néron model we have $A(\mathcal{L}) \cong \mathbf{A}_w(\mathcal{O}_w)$. Thus we have a reduction map $\mathrm{red}_w : A(\mathcal{L}) \to \tilde{\mathbf{A}}_w(k^s)$ and we denote $\tilde{y} := \mathrm{red}_w(y)$. The restriction σ_w of σ to $\mathcal{L}$ is an element of the inertia group $I(w|v)$ of w over v. Notice that $\mathrm{Gal}(\mathcal{L}/\mathcal{K}_v)$ is equal to the decomposition group $D(w|v)$ of w over v. In fact, this group coincides with $I(w|v)$, because $\kappa_w = \kappa_v = k^s$, where κ_w, resp. κ_v denotes the residue field of w, resp. v. Since the reduction $\tilde{\sigma}_w$ of σ_w modulo w is trivial and red_w commutes with the action of $\mathrm{Gal}(\mathcal{L}/\mathcal{K}_v)$, then $\widetilde{\sigma_w(y)} = \tilde{\sigma}_w(\tilde{y}) = \tilde{y}$. But by [BoLuRa90, lemma 2 (b), chapter 7, §7.3] the multiplication by p^n is étale in $\mathbf{A}_w$. Thus red_w is injective in $A(\mathcal{L}) \cap A[p^n]$, whence $\sigma(y) = \sigma_w(y) = y$ for every $v \in \mathcal{U}_s$ and $\sigma \in I_v$, *i.e.*, $y \in A(\mathcal{K}^{\mathrm{ur}})$. □

Remark 2.2. Note that the exact sequence (2.1.1) remains exact after passing to the quotient by $\tau B(k^s)$, thus we obtain for every $n \geq 1$ the exact sequence of groups

$$0 \to \left(\frac{A(\mathcal{K}^{\mathrm{ur}})}{\tau B(k^s)}\right)[p^n] \to \frac{A(\mathcal{K}^{\mathrm{ur}})}{\tau B(k^s)} \xrightarrow{\times p^n} \frac{A(\mathcal{K}^{\mathrm{ur}})}{\tau B(k^s)} \to 0, \tag{2.2.1}$$

where the first group of (2.2.1) denotes the p^n-torsion subgroup of $A(\mathcal{K}^{\mathrm{ur}})/\tau B(k^s)$. Moreover, the long exact Galois cohomology sequence derived from (2.2.1) produces p^n-descent map

$$\delta_{p^n} : \frac{A(\mathcal{K})/\tau B(k^s)}{p^n(A(\mathcal{K})/\tau B(k^s))} \hookrightarrow H^1\left(\pi_1(\mathcal{U}_s, \eta_s), \left(\frac{A(\mathcal{K}^{\mathrm{ur}})}{\tau B(k^s)}\right)[p^n]\right)$$

for every $n \geq 1$. Since $(A(\mathcal{K})/\tau B(k^s))/p^n(A(\mathcal{K})/\tau B(k^s)) \cong (A(\mathcal{K})/\tau B(k^s)) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$, taking the injective limit we have a p^∞-descent map

$$\delta_{p^\infty} : \frac{A(\mathcal{K})}{\tau B(k^s)} \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^1\left(\pi_1(\mathcal{U}_s, \eta_s), \left(\frac{A(\mathcal{K}^{\mathrm{ur}})}{\tau B(k^s)}\right)[p^\infty]\right).$$

Remark 2.3. For each $v \in \mathcal{C}_s \setminus \mathcal{U}_s$, let $R_v \subset I_v$ be the first ramification group at v. Then R_v is the Sylow q-subgroup of I_v. Grothendieck showed in [Gr72, exp. 9, (4.6.3)] that if A has semi-stable reduction at some $v \in \mathcal{C}_s$, then the Swan conductor $\delta_{p,v}(A)$ of A at v with respect to a prime number $p \neq q$ vanishes. This is equivalent to R_v acting trivially on $A[p]$. Since for every $v \in \mathcal{U}_s$, the group I_v acts trivially on $A[p]$, we conclude that $\delta_{p,v}(A) = 0$ for every $v \in \mathcal{C}_s$ if and only if $\mathcal{K}(A[p])/\mathcal{K}$ is unramified at every $v \in \mathcal{U}_s$ and at most tamely ramified at every $v \in \mathcal{C}_s \setminus \mathcal{U}_s$, *i.e.*, $\mathcal{K}(A[p]) \subset \mathcal{K}^t$.

If $q = 0$ or $q > 2d + 1$, Ogg observed in [Og62, remark 1, p. 211] there exists a prime number $p \neq q$ such that $\mathcal{K}(A[p])/\mathcal{K}$ is tamely ramified. Thus $\delta_{p,v}(A) = 0$ for every $v \in \mathcal{C}_s$. But Grothendieck also showed in [Gr72, exp. 9, corollaire 4.6] that $\delta_{p,v}(A)$ does not depend on p. In particular, $\mathcal{K}(A[p])/\mathcal{K}$ is tamely ramified for every prime number $p \neq q$. Furthermore, Deschamps proved in [De81, corollaire 5.18] that A acquires semi-stable reduction over $\mathcal{K}(A[p])$.

Hypothesis 2.4. Summing-up, we assume from now on that $q = 0$ or $q > 2d + 1$, so that A acquires semi-stable reduction over a finite tamely ramified extension of $\mathcal{K}$, namely $\mathcal{K}(A[p])$ for any prime number $p \neq q$. We also fix from now on a prime number $p \neq q$.

Lemma 2.5. *For every $n \geq 1$ and $v \in \mathcal{C}_s \setminus \mathcal{U}_s$, the ramification group R_v acts trivially on $A[p^n]$. Moreover, we have an exact sequence of groups*

$$0 \to A[p^n] \to A(\mathcal{K}^t) \xrightarrow{\times p^n} A(\mathcal{K}^t) \to 0. \tag{2.5.1}$$

Proof. We proceed similarly to the proof of lemma 2.1. We show the first statement by induction. Let $y \in A[p^2]$, $x := py \in A[p]$, $v \in \mathcal{C}_s \setminus \mathcal{U}_s$ and $\sigma \in R_v$. Then $\sigma(py) = p\sigma(y) = \sigma(x) = x = py$, *i.e.*, $\sigma(y) - y \in A[p]$.

Let $\mathcal{L} := \mathcal{K}_v((1/p)x)$ be the Galois subextension of $\mathcal{K}^s_{v^s}/\mathcal{K}_v$ generated by the solutions $y' \in A(\mathcal{K}^s_{v^s})$ of $py' = x$ over $\mathcal{K}_v$. Let w be the extension of v to $\mathcal{L}$. Let $\tilde{\mathbf{A}}^0_w$ be the connected component of the special fiber $\tilde{\mathbf{A}}_w$ of the Néron minimal model $\mathbf{A}_w$ of A over $\mathcal{O}_w$. Again the reduction $\tilde{\sigma}_w$ of the restriction σ_w of σ to $\mathcal{L}$ modulo w is trivial, thus $\widetilde{\sigma_w(y)} = \tilde{\sigma}_w(\tilde{y}) = \tilde{y}$. Once more by [BoLuRa90, lemma 2 (b), chapter 7, §7.3] the map $\mathrm{red}_w : A(\mathcal{L}) \to \tilde{\mathbf{A}}^0_w(k^s)$ is injective in $A(\mathcal{L}) \cap A[p] \subset A(\mathcal{K}^t)$, whence $\sigma(y) = \sigma_w(y) = y$ for every $v \in \mathcal{C}_s \setminus \mathcal{U}_s$ and $\sigma \in R_v$. We have already observed in the proof of lemma 2.1 that $\sigma(y) = y$ for every $v \in \mathcal{U}_s$ and $\sigma \in I_v$. Consequently $y \in A(\mathcal{K}^t)$. Now the first claim follows by induction.

For the proof of the surjectivity of the multiplication by p^n map follows exactly the proof of lemma 2.1, once we know that for every $v \in \mathcal{C}_s \setminus \mathcal{U}_s$ and $\sigma \in R_v$, σ acts trivially on $A[p^n]$; and similarly for every $v \in \mathcal{U}_s$ and $\sigma \in I_v$, σ acts trivially on $A[p^n]$. $\square$

Remark 2.6. As before, the exact sequence (2.5.1) remains also exact after taking the quotient by $\tau B(k^s)$ leading to the exact sequence

$$0 \to \left(\frac{A(\mathcal{K}^t)}{\tau B(k^s)}\right)[p^n] \to \frac{A(\mathcal{K}^t)}{\tau B(k^s)} \xrightarrow{\times p^n} \frac{A(\mathcal{K}^t)}{\tau B(k^s)} \to 0. \tag{2.6.1}$$

The long exact Galois cohomology sequence derived from (2.6.1) shows that δ_{p^n} factors through the tame p^n-descent map

$$\delta^t_{p^n} : \frac{A(\mathcal{K})/\tau B(k^s)}{p^n(A(\mathcal{K})/\tau B(k^s))} \hookrightarrow H^1\left(\pi^t_1(\mathcal{U}_s, \eta_s), \left(\frac{A(\mathcal{K}^t)}{\tau B(k^s)}\right)[p^n]\right).$$

Taking the injective limit we obtain a tame p^∞-descent map

$$\delta^t_{p^\infty} : \frac{A(\mathcal{K})}{\tau B(k^s)} \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^1\left(\pi^t_1(\mathcal{U}_s, \eta_s), \left(\frac{A(\mathcal{K}^t)}{\tau B(k^s)}\right)[p^\infty]\right).$$

We end this section with following two observations.

Lemma 2.7. *The subgroup of the elements of $(A(\mathcal{K}^t)/\tau B(k^s))[p^\infty]$ which are fixed under the action of $\pi^t_1(\mathcal{U}_s, \eta_s)$ is finite.*

Proof. Note that the set of elements of $(A(\mathcal{K}^t)/\tau B(k^s))[p^\infty]$ fixed by $\pi^t_1(\mathcal{U}_s, \eta_s) \cong \mathrm{Gal}(\mathcal{K}^t/\mathcal{K})$ is simply $(A(\mathcal{K})/\tau B(k^s))[p^\infty]$ which is finite by the Lang-Néron theorem. $\square$

Remark 2.8. Observe that for every $n \geq 1$ we have

$$\left(\frac{A(\mathcal{K}^t)}{\tau B(k^s)}\right)[p^n] = \frac{A[p^n] + \tau B(k^s)}{\tau B(k^s)} \cong \frac{A[p^n]}{\tau B[p^n]}. \tag{2.8.1}$$

Denote $A[p^\infty]/\tau B[p^\infty] := \varinjlim_n A[p^n]/\tau B[p^n]$. As a consequence we rewrite the tame p^∞-descent map as

$$\delta^t_{p^\infty} : \frac{A(\mathcal{K})}{\tau B(k^s)} \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^1\left(\pi_1^t(\mathcal{U}_s, \eta_s), \frac{A[p^\infty]}{\tau B[p^\infty]}\right). \tag{2.8.2}$$

3. Selmer groups

Definition 3.1. Let $j : \eta_s \hookrightarrow \mathcal{C}_s$ be the inclusion map. For each discrete p-primary torsion étale sheaf F in η_s, let $\mathcal{F} := j_*F$. The Selmer group of $\mathcal{C}$ with respect to F, denoted by $S(\mathcal{C}, F)$, is defined as the first étale cohomology group $H^1_{\text{ét}}(\mathcal{C}_s, \mathcal{F})$ of $\mathcal{C}_s$ with coefficients in $\mathcal{F}$. Consider the p-primary étale sheaf $F_{p^\infty} := A[p^\infty]/\tau B[p^\infty]$ in η_s. Let $\mathcal{F}_{p^\infty} := j_*(F_{p^\infty})$. In the abstract, we denoted $S(\mathcal{C}, F_{p^\infty})$ by $\mathrm{Sel}_p(A/K)$.

Remark 3.2. Using some results on étale cohomology (*cf.* [Mi80, III.1.25, V.2.17]), the Selmer group $S(\mathcal{C}, F)$ can be alternatively described in terms of Galois cohomology by

$$S(\mathcal{C}, F) = \mathrm{Ker}\left(H^1(\pi_1^t(\mathcal{U}_s, \eta_s), F) \to \bigoplus_{v \in \mathcal{C}_s \setminus \mathcal{U}_s} H^1(\mathrm{Gal}(\mathcal{K}_v^t/\mathcal{K}_v), F)\right), \tag{3.2.1}$$

where $\mathcal{K}_v^t/\mathcal{K}_v$ is the maximal Galois subextension of $\mathcal{K}_{v^s}^s/\mathcal{K}_v$ tamely ramified over $\mathcal{K}_v$. In the case where $F = F_{p^\infty}$, this definition also agrees with the classical one for $S(\mathcal{C}, F_{p^\infty})$. First observe that the local tame p^∞-descent map

$$\delta^t_{p^\infty, v} : \frac{A(\mathcal{K}_v)}{\tau B(k^s)} \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^1(\mathrm{Gal}(\mathcal{K}_v^t/\mathcal{K}_v), F_{p^\infty})$$

has trivial image. Indeed, by a result of Mattuck [Ma55], $A(\mathcal{K}_v)$ is isomorphic to $\mathcal{O}_v^d$ (as an additive group) times a finite group, where $\mathcal{O}_v$ denotes the ring of integers of $\mathcal{K}_v$. Since $\mathrm{char}(\mathcal{K}_v) = q \neq p$, then $(A(\mathcal{K}_v)/\tau B(k^s)) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p = 0$.

Before we proceed, let us recall the definition of the conductor $\mathfrak{F}_A$ of the abelian variety A/K of dimension d. Let $T_p(A)$ the p-adic Tate module of A and $V_p(A) := T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Let $\mathcal{X}_s := \mathcal{X} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^s)$.

Definition 3.3. The multiplicity e_v of the conductor $\mathfrak{F}_A$ of A at $v \in \mathcal{X}_s$ is defined as $\mathrm{codim}(V_p(A)^{I_v})$, where $V_p(A)^{I_v}$ denotes the subspace of $V_p(A)$ which is fixed by the action of the inertia group I_v. The multiplicity e_v can be described in terms of the type of the reduction of A at v as follows. Let $\mathbf{A}_v \to \mathrm{Spec}(\mathcal{O}_v)$ be the Néron minimal model of A at $v \in \mathcal{X}$, $\tilde{\mathbf{A}}_v \to \mathrm{Spec}(k^s)$ its special fiber and $\tilde{\mathbf{A}}_v^0$ the connected component of $\tilde{\mathbf{A}}_v$. Then $\tilde{\mathbf{A}}_v^0$ is an extension of an abelian variety $\mathfrak{A}_v$ by a linear algebraic group L_v both defined over k^s. Let $a_v := \dim(\mathfrak{A}_v)$. The linear algebraic group L_v is equal to a product $\mathbb{G}_m^{t_v} \times \mathbb{G}_a^{u_v}$. The non-negative integers a_v, t_v and u_v are called the *abelian*, *reductive* and *unipotent* ranks of $\tilde{\mathbf{A}}_v^0$, resp.. Since we are supposing that k has characteristic either 0 or greater than $2d+1$, we have no contribution from the Swan conductor, in this circumstance the multiplicity e_v of $\mathfrak{F}_A$ at v equals $t_v + 2u_v$. Recall also that $d = a_v + t_v + u_v$.

Let M be a $\mathbb{Z}_p$-module and $M^* := \mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ its PONTRYAGIN dual module. If M^* is finitely generated, then we say that M is *cofinitely generated*. In this case the rank of M^* is called the corank of M denoted by $\mathrm{cork}_{\mathbb{Z}_p}(M)$. We say that M is a $\mathbb{Z}_p$-*cofree* module, if M^* is a $\mathbb{Z}_p$-free module.

Lemma 3.4. *Let*

$$\mathcal{H}(\mathcal{C}, F_{p^\infty}) := \bigoplus_{v \in \mathcal{C}_s \setminus \mathcal{U}_s} H^1(\mathrm{Gal}(\mathcal{K}_v^t/\mathcal{K}_v), F_{p^\infty}).$$

Then $\mathcal{H}(\mathcal{C}, F_{p^\infty})$ is a $\mathbb{Z}_p$-cofree module of corank equal to

$$\sum_{v \in \mathcal{C}_s \setminus \mathcal{U}_s} (2a_v + t_v) - \#(\mathcal{C}_s \setminus \mathcal{U}_s) \cdot (2d_0).$$

Proof. Note that $\mathrm{Gal}(\mathcal{K}_v^t/\mathcal{K}_v) \cong I_v/R_v$ is procyclic. It follows from [NeScWi00, 1.6.13 (i), p. 69] that

$$H^1(\mathrm{Gal}(\mathcal{K}_v^t/\mathcal{K}_v), F_{p^\infty}) \cong (F_{p^\infty})_{\mathrm{Gal}(\mathcal{K}_v^t/\mathcal{K}_v)},$$

where $(F_{p^\infty})_{\mathrm{Gal}(\mathcal{K}_v^t/\mathcal{K}_v)}$ denotes the subspace of F_{p^∞} of the elements which are coinvariant with respect to the action of $\mathrm{Gal}(\mathcal{K}_v^t/\mathcal{K}_v)$. But, by lemma 2.5, R_v acts as identity on $A[p^\infty]$, therefore

$$(F_{p^\infty})_{\mathrm{Gal}(\mathcal{K}_v^t/\mathcal{K}_v)} \cong (F_{p^\infty})_{I_v} \cong (A[p^\infty])_{I_v}/\tau B[p^\infty].$$

The latter is $\mathbb{Z}_p$-cofree with $\mathbb{Z}_p$-corank equal to $2d - e_v - 2d_0 = 2a_v + t_v - 2d_0$ and this implies the lemma. □

Definition 3.5. Let $\mathfrak{F}_{A,\mathcal{C}} := \sum_{v \in \mathcal{C}} e_v v$ be the restriction of the conductor $\mathfrak{F}_A$ of A to $\mathcal{C}$ and $f_{A,\mathcal{C}} := \deg(\mathfrak{F}_{A,\mathcal{C}})$.

Proposition 3.6. *(cf.* [El06, proposition 2.5]*)*

(1) *$H^1(\pi_1^t(\mathcal{U}_s, \eta_s), F_{p^\infty})$ is a cofree $\mathbb{Z}_p$-module of corank equal to*

$$(2d - 2d_0)(2g - 2 + \#(\mathcal{X}_s \setminus \mathcal{C}_s)) + f_{A,\mathcal{C}} + \sum_{v \in \mathcal{C}_s \setminus \mathcal{U}_s} (2a_v + t_v) - \#(\mathcal{C}_s \setminus \mathcal{U}_s) \cdot (2d_0).$$

(2) *$S(\mathcal{C}, F_{p^\infty})$ is a $\mathbb{Z}_p$-module of corank equal to*

$$(2d - 2d_0)(2g - 2 + \#(\mathcal{X}_s \setminus \mathcal{C}_s)) + f_{A,\mathcal{C}}.$$

Proof. The proof is similar to [El06, proposition 2.5], replacing [El06, remark 2.4] by lemma 2.7, taking into account that [Mi80, chapter V, (2.18)] yields that $H^1(\pi_1^t(\mathcal{U}_s, \eta_s), F_{p^\infty})$ has $\mathbb{Z}_p$-corank equal to

$$\begin{aligned}&(2d - 2d_0)(2g - 2 + \#(\mathcal{X}_s \setminus \mathcal{U}_s)) \\ &= (2d - 2d_0)(2g - 2 + \#(\mathcal{X}_s \setminus \mathcal{C}_s)) + \sum_{v \in \mathcal{C}_s \setminus \mathcal{U}_s} (2a_v + t_v) + f_{A,\mathcal{C}} - \#(\mathcal{C}_s \setminus \mathcal{U}_s) \cdot (2d_0).\end{aligned}$$

and using lemma 3.4. □

The framework of lemma 3.4 and proposition 3.6 allows one to immediately extend [El06, theorem 2.8] to abelian varieties as follows.

Definition 3.7. [El06, definition 2.6] Let G be a finite group, H a subgroup of $\mathrm{Aut}(G)$ and $\Gamma := G \rtimes H$ the semi-direct product of G and H. Denote by V_Γ, resp. V_G, the real vector space spanned by the irreducible complex-valued characters of Γ, resp. G. An element $v \in V_\Gamma$, resp. V_G, is non-negative if $\langle v, \psi \rangle \geq 0$ for every character ψ of an irreducible representation of Γ, resp. G. Denote by $[\Gamma/H] \in V_\Gamma$ the coset character of Γ with respect to H and by $[G/1] \in V_G$ the regular character of G. Let $\epsilon(G, H)$ be the maximum of $\langle v, [\Gamma/H]\rangle$ over all $v \in V_\Gamma$ such that

(1) $v \geq 0$.
(2) $[G/1] - r(v) \geq 0$, where $r : V_\Gamma \to V_G$ is the restriction map.

This number is well-defined, since the region of V_Γ defined by the two previous conditions is a compact polytope.

Let $\psi : \mathcal{C}' \to \mathcal{C}$ a finite morphism defined over k which is geometrically Galois. Let l/k be the smallest finite Galois extension of k such that all elements of $\mathcal{G} = \mathrm{Aut}_{\overline{k}}(\mathcal{C}'/\mathcal{C})$ are defined over l. Hence, G_k acts on $\mathcal{G}$ via the finite quotient $H := \mathrm{Gal}(l/k)$.

Theorem 3.8. *(cf.* [El06, theorem 2.8]*) Let $\mathcal{C}/k$ be a smooth geometrically connected curve of genus g defined over a field k with function field $K = k(\mathcal{C})$. Let A/K be a non constant abelian variety of dimension d. Suppose that k has characteristic either 0 or greater than $2d + 1$. Let d_0 be the dimension of the K/k-trace of A and $\mathcal{X}$ a regular compactification of $\mathcal{C}$. Denote by k^s a separable closure of k, and $\mathcal{C}_s := \mathcal{C} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^s)$ and $\mathcal{X}_s := \mathcal{X} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^s)$. Let $f_{A,\mathcal{C}}$ be the degree of the conductor of A with respect to $\mathcal{C}$. Let $\psi : \mathcal{C}' \to \mathcal{C}$ be a finite morphism defined over k which is geometrically* Galois *and étale with automorphism group $\mathcal{G} = \mathrm{Aut}_{k^s}(\mathcal{C}'/\mathcal{C})$. Then*

$$\text{(3.8.1)} \qquad \mathrm{rank}\left(\frac{A(K')}{\tau' B'(k)}\right) \leq \epsilon(\mathcal{G}, H)((2d - 2d_0)(2g - 2 + \#(\mathcal{X}_s \setminus \mathcal{C}_s)) + f_{A,\mathcal{C}}).$$

Proof. The proof is the same as in [El06, theorem 2.8] replacing [El06, lemma 2.9] by lemma 3.9. □

Let $\mathcal{A}' := \mathcal{A} \times_{\mathcal{C}} \mathcal{C}'$ and $\phi' : \mathcal{A}' \to \mathcal{C}'$ the morphism obtained from ϕ by changing the base from $\mathcal{C}$ to $\mathcal{C}'$. Denote by $\mathcal{U}'$ the smooth locus of ϕ' and let $\mathcal{U}'_s := \mathcal{U}' \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^s)$. Let η'_s be a geometric generic point of $\mathcal{U}'$ and $\pi_1^t(\mathcal{U}'_s, \eta'_s)$ the tame algebraic fundamental group of $\mathcal{U}'_s$ with respect to η'_s. Let $K' := k(\mathcal{C}')$ and denote by (τ', B') the K'/k-trace of A. Let $d'_0 := \dim(B')$. Let $F'_{p^\infty} := A[p^\infty]/\tau' B'[p^\infty]$.

Lemma 3.9. *(cf.* [El06, lemma 2.9]*) With the same hypotheses of theorem 3.8,*

$$\mathrm{Hom}(S(\mathcal{C}', F'_{p^\infty}), \mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is a free $\mathbb{Q}_p[\mathcal{G}]$-module of rank at most

$$(2d - 2d_0)(2g - 2 + \#(\mathcal{X}_s \setminus \mathcal{C}_s)) + f_{A,\mathcal{C}}.$$

Proof. For any discrete cofinitely generated $\mathbb{Z}_p[\mathcal{G}]$-module M we associate the finitely generated $\mathbb{Q}_p[\mathcal{G}]$-module $W(M) := \mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Let $\mathcal{L} := k^s(\mathcal{C}')$, $w \in \mathcal{C}'_s := \mathcal{C}' \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^s)$ and $\mathcal{L}_w$ the completion of $\mathcal{L}$ at w. Let $\mathcal{L}^s$ be a separable closure of $\mathcal{L}$, w^s the extension of w to $\mathcal{L}^s$ and $\mathcal{L}^s_{w^s}$ the completion of $\mathcal{L}^s$ at w^s. Let $\mathcal{L}^t_w/\mathcal{L}_w$ be the maximal Galois subextension of $\mathcal{L}^s_{w^s}/\mathcal{L}_w$ which is tamely ramified over $\mathcal{L}_w$.

As before, we consider the following $\mathbb{Z}_p[\mathcal{G}]$-module

$$\mathcal{H}(\mathcal{C}', F'_{p^\infty}) := \bigoplus_{w \in \mathcal{C}'_s \setminus \mathcal{U}'_s} H^1(\mathrm{Gal}(\mathcal{L}^t_w/\mathcal{L}_w), F'_{p^\infty}).$$

It follows from the proof of lemma 3.4 that for each $w \in \mathcal{C}'_s \setminus \mathcal{U}'_s$ the $\mathbb{Z}_p$-module $H^1(\mathrm{Gal}(\mathcal{L}^t_w/\mathcal{L}), F'_{p^\infty})$ has $\mathbb{Z}_p$-corank $2a_w + t_w - 2d'_0$. Since $\mathcal{C}' \to \mathcal{C}$ is geometrically étale, the morphism $\mathbf{A}_v \times_{\mathrm{Spec}(\mathcal{O}_v)} \mathrm{Spec}(\mathcal{O}_w) \to \mathbf{A}_w$ of change of base of NÉRON models is an isomorphism (*cf.* [BoLuRa90, chapter 7, theorem 1, p. 176]). *A fortiori*, $a_w = a_v$, $t_w = t_v$ and $u_w = u_v$ for every $w \in \mathcal{C}'_s$ over $v \in \mathcal{C}_s$. As a consequence, $W(\mathcal{H}(\mathcal{C}', F'_{p^\infty}))$ is a free $\mathbb{Q}_p[\mathcal{G}]$-module of rank equal to

$$\sum_{v \in \mathcal{C}_s \setminus \mathcal{U}_s} (2a_v + t_v) - \#(\mathcal{C}_s \setminus \mathcal{U}_s) \cdot (2d'_0). \tag{3.9.1}$$

Let $j' : \eta'_s \hookrightarrow \mathcal{C}'_s$ be the inclusion map and $\mathcal{F}'_{p^\infty} := j'_*(F'_{p^\infty})$. As in [El06, proof of proposition 2.5] the finiteness of $H^2_{\text{ét}}(\mathcal{C}'_s, \mathcal{F}'_{p^\infty})$ implies the equality

$$[W(H^1(\pi^t_1(\mathcal{U}'_s, \eta'_s), F'_{p^\infty}))] = [W(S(\mathcal{C}', F'_{p^\infty}))] + [W(\mathcal{H}(\mathcal{C}', F'_{p^\infty}))] \tag{3.9.2}$$

in the GROTHENDIECK group of the category of $\mathbb{Q}_p[\mathcal{G}]$-modules. It follows from SHAPIRO's lemma [Se86, proposition 10, p. I-12] that

$$H^1(\pi^t_1(\mathcal{U}'_s, \eta'_s), F'_{p^\infty}) = H^1(\pi^t_1(\mathcal{U}_s, \eta_s), F'_{p^\infty} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathcal{G}]).$$

By [Mi80, chapter V, remark 2.19], for every $\pi^t_1(\mathcal{U}_s, \eta_s)$-module M we have an identity

$$\begin{aligned}[H^1(\pi^t_1(\mathcal{U}_s, \eta_s), M)] - [H^0(\pi^t_1(\mathcal{U}_s, \eta_s), M)]& \\ = (2g - 2 + \#(\mathcal{X}_s \setminus \mathcal{U}_s))[M].&\end{aligned} \tag{3.9.3}$$

But the previous construction is functorial, so we can view (3.9.3) as an equality in the GROTHENDIECK group of cofinitely generated $\mathbb{Z}_p[\mathcal{G}]$-modules when $M = F'_{p^\infty} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathcal{G}]$. Once again using SHAPIRO's lemma and lemma 2.7 for the curve $\mathcal{U}'$ we conclude that $H^0(\pi^t_1(\mathcal{U}_s, \eta_s), F'_{p^\infty} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathcal{G}]) = H^0(\pi^t_1(\mathcal{U}'_s, \eta'_s), F'_{p^\infty})$ is finite, hence $\mathbb{Z}_p$-cotorsion, in particular it is killed by the functor W. It then follows from (3.9.3), (3.9.1) and (3.9.2) that $W(S(\mathcal{C}', F'_{p^\infty})$ is a $\mathbb{Q}_p[\mathcal{G}]$-free module of rank equal to

$$\begin{aligned}(2d - 2d'_0)&(2g - 2 + \#(\mathcal{X}_s \setminus \mathcal{U}_s)) \\ &- \sum_{v \in \mathcal{C}_s \setminus \mathcal{U}_s} (2a_v + t_v) + \#(\mathcal{C}_s \setminus \mathcal{U}_s) \cdot (2d'_0) = \\ &(2d - 2d'_0)(2g - 2 + \#(\mathcal{X}_s \setminus \mathcal{C}_s)) \\ &+ \sum_{v \in \mathcal{C}_s \setminus \mathcal{U}_s} (2a_v + t_v) + f_{A,\mathcal{C}} - \#(\mathcal{C}_s \setminus \mathcal{U}_s) \cdot (2d'_0) \\ &- \sum_{v \in \mathcal{C}_s \setminus \mathcal{U}_s} (2a_v + t_v) + \#(\mathcal{C}_s \setminus \mathcal{U}_s)(2d'_0) \\ &\leq (2d - 2d_0)(2g - 2 + \#(\mathcal{X}_s \setminus \mathcal{C}_s)) + f_{A,\mathcal{C}}.\end{aligned} \tag{3.9.4}$$

In order to see the validity of the latter inequality in (3.9.4) we use the fact that the trace maps $\tau : B \to A$ and $\tau' : B' \to A$ are injective, if k has characteristic zero, and have finite kernel if k has positive characteristic (for a discussion on this

cf. [Co05, §2 and §6]). In particular, the induced map $B \to B'$ has necessarily finite kernel, therefore $d_0 \leq d_0'$ and this implies the searched inequality. $\square$

Let $A' := A \times_K K'$, $\mathfrak{F}_{A'}$ the conductor divisor of A' on $\mathcal{X}'$ and $f_{A'} := \deg(\mathfrak{F}_{A'})$. Let $\mathfrak{F}_{A',\mathcal{C}'}$ be the restriction of $\mathfrak{F}_{A'}$ to $\mathcal{C}'$ and $f_{A',\mathcal{C}'} := \deg(\mathfrak{F}_{A',\mathcal{C}'})$.

As observed in [El06], when G is abelian and $H = \mathrm{Gal}(l/k)$, then $\epsilon(G,H) = \#\mathfrak{O}_{G_k}(G)$. Therefore, theorem 3.8 implies the following result.

Theorem 3.10. *(cf.* [El06, corollary 2.13]*) With the same notation of theorem 3.8, let A/K be a non constant abelian variety of dimension d. Suppose that k has either characteristic 0 or greater than $2d+1$. Let $\psi : \mathcal{C}' \to \mathcal{C}$ be a finite morphism defined over k which is geometrically abelian and étale with automorphism group $\mathcal{G} = \mathrm{Aut}_{k^s}(\mathcal{C}'/\mathcal{C})$. Denote by g' the genus of $\mathcal{C}'$, $\mathcal{X}'$ a regular compactification of $\mathcal{C}'$ and $f_{A',\mathcal{C}'}$ the conductor of $A' = A \times_K K'$ with respect to $\mathcal{C}'$. Then*

$$\text{(3.10.1)}\quad \operatorname{rank}\left(\frac{A(K')}{\tau' B'(k)}\right) \leq \frac{\#\mathfrak{O}_{G_k}(\mathcal{G})}{\#\mathcal{G}}((2d-2d_0)(2g'-2+\#(\mathcal{X}'_s \setminus \mathcal{C}'_s)) + f_{A',\mathcal{C}'}).$$

Remark 3.11. When $\mathcal{C} = \mathcal{X}$ is a complete curve and k is a number field, under the hypotheses of theorem 3.8, (3.10.1) improves [Pa05, (1.7)]. Observe that here we make no hypothesis concerning the truth of Tate's conjecture and nor the irreducibility of monodromy representations. Nevertheless, the method of [Pa05] allowed us to treat the case of arbitrary ramification.

We now compare the bound obtained here with that of [Pa05, (1.4)] in the ramified case. Let k be a number field, $\mathcal{X}/k$ be a complete geometrically connected smooth curve defined over k with function field $K := k(\mathcal{X})$. Let $\psi : \mathcal{X}' \to \mathcal{X}$ be a geometrically abelian cover defined over k. Let $\mathcal{R}$ be the ramification locus of ψ, $\mathcal{C} := \mathcal{X} \setminus \mathcal{R}$, $\mathcal{R}' := \psi^{-1}(\mathcal{R})$ and $\mathcal{C}' := \mathcal{X}' \setminus \mathcal{R}'$. Then the restriction of ψ to $\mathcal{C}'$ gives a geometrically abelian, now also geometrically étale, cover $\mathcal{C}' \to \mathcal{C}$ of affine curves defined over k. Let $\mathcal{R}_s := \mathcal{R} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^s) = \mathcal{C}_s \setminus \mathcal{U}_s$. Let $\mathfrak{F}_{A,\mathcal{R}} := \mathfrak{F}_A - \mathfrak{F}_{A,\mathcal{C}}$ and $f_{A,\mathcal{R}} := \deg(\mathfrak{F}_{A,\mathcal{R}})$.

In [Pa05, (1.4)], under the aforementioned hypotheses, the bound obtained was

$$\text{(3.11.1)}\qquad \frac{\#\mathfrak{O}_{G_k}(\mathcal{G})}{\#\mathcal{G}}(d(2d+1)(2g'-2)) + \#\mathfrak{O}_{G_k}(\mathcal{G}) \cdot 2df_A.$$

In particular, by [Pa05, proposition 3.7], (3.11.1) is greater or equal to

$$\text{(3.11.2)}\qquad \frac{\#\mathfrak{O}_{G_k}(\mathcal{G})}{\#\mathcal{G}}(d(2d+1)(2g'-2) + 2df_{A'}).$$

We now compare (3.11.2) with (3.10.1). We see that $d(2d+1)(2g'-2) \geq (2d - 2d_0)(2g'-2)$, if $g' \geq 1$. Clearly $2df_{A',\mathcal{C}'} \geq f_{A',\mathcal{C}'}$. All we need to analyze is when $2df_{A',\mathcal{R}'} \geq (2d-2d_0)\cdot\#\mathcal{R}'_s$. This inequality holds if and only if for every $w \in \mathcal{R}'_s$ we have $2de_w \geq 2d-2d_0$. The latter inequality holds as long as $e_w \geq 1$, *i.e.*, A' has bad reduction at w. In fact, otherwise $d_0 \geq d$, whence $d_0 = d$. Since the characteristic of k is zero, then the map $\tau : B \to A$ is injective, therefore A would be constant. However we are ruling out this possibility since the beginning. So theorem 3.10 gives a smaller bound for the rank of $A(K')/\tau' B'(k)$ than that of [Pa05, (1.7)] if

(a) $g' \geq 1$; and
(b) $\mathcal{R}' \subset \Delta'$, where Δ' denotes the discriminant locus of $\phi' : \mathcal{A}' \to \mathcal{C}'$.

4. Towers of function fields

Let $\mathcal{C}$ be a smooth geometrically connected curve defined over a field k of characteristic q which is either zero or greater than $2d+1$. We define a *tower of curves* over $\mathcal{C}$ to be a sequence

$$\mathcal{T} : \cdots \to \mathcal{C}_n \to \cdots \to \mathcal{C}_1 \to \mathcal{C}_0 := \mathcal{C}$$

of finite geometrically Galois and étale covers of curves $\mathcal{C}_n \to \mathcal{C}_0$ defined over k. For each cover $\mathcal{C}_n \to \mathcal{C}$ denote by $\mathcal{G}_n := \mathrm{Aut}_{k^s}(\mathcal{C}_n/\mathcal{C})$ the corresponding Galois group. The Galois group of the tower $\mathcal{T}$ is defined as $\mathcal{G}_\infty := \varprojlim_n \mathcal{G}_n$.

Let A/K be a non constant abelian variety. For each $n \geq 0$, let $K_n := k(\mathcal{C}_n)$ be the function field of $\mathcal{C}_n$ and (τ_n, B_n) be the K_n/k-trace of A.

When we consider the question of the size of the rank of Lang-Néron groups of abelian varieties over function fields we can ask the following vertical question : how does the rank of $A(K_n)/\tau_n B_n(k)$ vary along the tower $\mathcal{T}$?

In the case where $\mathcal{C}$ is a complete curve and k is a number field we studied this question in the special cases of the two particular towers:

(*) when $\mathcal{C}$ is an elliptic curve and $\mathcal{C}_n$ is obtained from $\mathcal{C}$ as the pull-back by the multiplication by n map in $\mathcal{C}$;

(**) for a curve $\mathcal{C}$ of any positive genus, $\mathcal{C}_n$ is obtained as the pull-back of $\mathcal{C}$ by the multiplication by n map in the Jacobian variety $J_\mathcal{C}$ of $\mathcal{C}$.

Observe that the first situation was already dealt with by Silverman in the case of elliptic fibrations in [Si02]. We proved in [Pa05, theorems 6.2 and 6.5] that average rank of $A(K_n)/\tau_n B_n(k)$ as $n \to \infty$ was smaller than a fixed multiple of the degree f_A of the conductor of A, under the hypothesis in (*) that $\mathcal{C}$ had no complex multiplication and in (**) that its Jacobian variety $J_\mathcal{C}$ had $\overline{k}$-endomorphism ring equal to $\mathbb{Z}$ (plus additional hypotheses, *cf.* [Pa05, theorem 6.5]).

Let $K_\infty := \varinjlim_n K_n$ and $\tau_\infty B_\infty(k) := \varinjlim_n \tau_n B_n(k)$. One may naturally ask whether the abelian group $A(K_\infty)/\tau_\infty B_\infty(k)$ is finitely generated.

In [El06], Ellenberg considered this question in the case of a non constant elliptic curve E/K and supposed that each $\mathcal{G}_n$ was isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ so that $\mathcal{G}_\infty \cong \mathbb{Z}_p$. For each $n \geq 0$, let $\mathcal{K}_n := k^s(\mathcal{C}_n)$ and $\mathcal{K}_\infty := \varinjlim_n \mathcal{K}_n$. Then under certain conditions [El06, theorem 4.4] he proved that $E(\mathcal{K}_\infty)$ is finitely generated. The goal of this section is to extend this result to the case of an abelian fibration.

We fix a prime number p different from q. We suppose that each $\mathcal{G}_n$ is a finite p-group, whence $\mathcal{G}_\infty$ is a pro-p group. For every $n \geq 0$, let $F_{n,p^\infty} := A[p^\infty]/\tau_n B_n[p^\infty]$, $\tau_\infty B_\infty[p^\infty] := \varinjlim_n \tau_n B_n[p^\infty]$ and $F_{\infty,p^\infty} := A[p^\infty]/\tau_\infty B_\infty[p^\infty]$. Let $S(\mathcal{C}_\infty, F_{\infty,p^\infty}) := \varinjlim_n S(\mathcal{C}_n, F_{n,p^\infty})$. Then $S(\mathcal{C}_\infty, F_{\infty,p^\infty})$ is a discrete p-primary group with a continuous action of $\mathcal{G}_\infty$. Hence it also comes equipped with an action of the Iwasawa algebra $\Lambda(\mathcal{G}_\infty) := \varprojlim_\mathcal{H} \mathbb{Z}_p[\mathcal{G}_\infty/\mathcal{H}]$, where $\mathcal{H}$ runs through the open normal subgroups of $\mathcal{G}_\infty$. For every étale sheaf $\mathcal{F}$ on $\mathcal{C}$, given an integer $n \geq 1$, denote by $\mathcal{F}_{|\mathcal{C}_n}$ the pull-back of $\mathcal{F}$ to $\mathcal{C}_n$ and let $H^i_{\text{ét}}(\mathcal{C}_{s,\infty}, \mathcal{F}) := \varinjlim_n H^i_{\text{ét}}(\mathcal{C}_{s,n}, \mathcal{F}_{|\mathcal{C}_n})$.

Hypothesis 4.1. Assume that $\mathcal{G}_\infty$ is a non trivial pro-p finite dimensional p-adic Lie group without p-torsion elements.

Under the hypothesis 4.1, the Iwasawa algebra $\Lambda(\mathcal{G}_\infty)$ is both right and left noetherian local ring without zero divisors. Moreover, it follows from [Ho02, lemma

1.6] that for every cofinitely generated discrete $\Lambda(\mathcal{G}_\infty)$-module M the GALOIS cohomology group $H^i(\mathcal{G}_\infty, M)$ is a cofinitely generated $\mathbb{Z}_p$-module. Furthermore, it makes sense to define the $\Lambda(\mathcal{G}_\infty)$-corank of M as follows [Ho02]

$$\operatorname{corank}_{\Lambda(\mathcal{G}_\infty)}(M) := \sum_{i\geq 0}(-1)^i \operatorname{corank}_{\mathbb{Z}_p}(H^i(\mathcal{G}_\infty, M)).$$

Lemma 4.2. *The $\Lambda(\mathcal{G}_\infty)$-module $S(\mathcal{C}_\infty, F_{\infty,p^\infty})$ is cofinitely generated and has $\Lambda(\mathcal{G}_\infty)$-corank equal to $(2d-2d_0)(2g-2+\#(\mathcal{X}_s \setminus \mathcal{C}_s)) + f_{A,\mathcal{C}}$.*

Proof. The proof follows as in [El06, propositions 3.3 and 3.4] replacing [El06, proposition 2.5] by proposition 3.6. Observe though that in the course of the proof of [El06, proposition 3.4] it is necessary to have $E[p^\infty]^{\pi_1(\mathcal{U}_s,\eta_s)}$ finite. In his case this followed from [El06, remark 2.4], in the current situation this follows from lemma 2.7. □

We assume from now on that $\mathcal{G}_\infty \cong \mathbb{Z}_p$. Let $\mathcal{H}_\infty := \operatorname{Ker}(\pi_1^t(\mathcal{U}_s, \eta_s) \twoheadrightarrow \mathcal{G}_\infty)$.

Proposition 4.3. *If k is finitely generated over its prime field κ_0, then the subspace $F_{0,p^\infty}^{\mathcal{H}_\infty}$ of F_{0,p^∞} of the elements fixed under the action of $\mathcal{H}_\infty$ is finite.*

Proof. Assume first that k is a finite field of characteristic q and let Frob_k be its FROBENIUS automorphism. Let $\pi_1^t(\mathcal{U}, \eta_s)$ be the *arithmetic tame fundamental group* of $\mathcal{U}$ with respect to η_s. By the definition of the $\mathcal{H}_\infty$ we have a short exact sequence of groups

$$1 \to \mathcal{H}_\infty \to \pi_1^t(\mathcal{U}_s, \eta_s) \to \mathcal{G}_\infty \to 1. \tag{4.3.1}$$

By the definition of $\pi_1^t(\mathcal{U}, \eta_s)$, there is also another sequence

$$1 \to \pi_1^t(\mathcal{U}_s, \eta_s) \to \pi_1^t(\mathcal{U}, \eta_s) \to G_k \to 1. \tag{4.3.2}$$

Since Frob_k acts on $\mathcal{H}_\infty$, both sequences (4.3.1) and (4.3.2) yield a third exact sequence

$$1 \to \mathcal{H}_\infty \to \pi_1^t(\mathcal{U}, \eta_s) \to \mathcal{G}_\infty \ltimes G_k \to 1. \tag{4.3.3}$$

Suppose that $F_{0,p^\infty}^{\mathcal{H}_\infty}$ is infinite. Let $V := \operatorname{Hom}(F_{0,p^\infty}^{\mathcal{H}_\infty}, \mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. This is a finite positive dimensional $\mathbb{Q}_p$-vector space endowed with an action of $\mathcal{G}_\infty \ltimes G_k$. The group $\mathcal{G}_\infty$ acts on V through its inclusion in $\mathcal{G}_\infty \ltimes G_k$. Since $\mathcal{G}_\infty$ is abelian, then V decomposes as a $\mathbb{Q}_p[\mathcal{G}_\infty]$-module into a sum of one dimensional eigenspaces.

Let $\chi : \mathcal{G}_\infty \to \mathbb{Q}_p^*$ be a character of $\mathcal{G}_\infty$ and V_χ the eigensubspace of V corresponding to χ. Let $V_\chi^{\mathrm{Frob}_k}$ be the subspace of V_χ of those elements which are fixed by Frob_k. Since the action of $\sigma \in \mathcal{G}_\infty$ on V_χ is through multiplication by $\chi(\sigma)$, then $V_\chi^{\mathrm{Frob}_k} = V_{\chi^q}$. Similarly, for every integer $n \geq 1$, we have $V_\chi^{\mathrm{Frob}_k^n} = V_{\chi^{q^n}}$. But V is finite dimensional, therefore there exists an integer $f \geq 1$ such that $\chi^{q^f} = \chi$, *i.e.*, χ^{q^f-1} is the trivial character. However, $\mathcal{G}_\infty \cong \mathbb{Z}_p$ is free, thus χ itself must be trivial. In particular, $\mathcal{G}_\infty$ acts trivially on V, whence on $F_{0,p^\infty}^{\mathcal{H}_\infty}$. In particular, the action of $\mathcal{G}_\infty \ltimes G_k$ on $F_{0,p^\infty}^{\mathcal{H}_\infty}$ reduces to that of G_k. Observe that by (4.3.2) this only happens if and only if $F_{0,p^\infty}^{\mathcal{H}_\infty} = (F_{0,p^\infty}^{\mathcal{H}_\infty})^{\pi_1^t(\mathcal{U}_s,\eta_s)} \subset F_{0,p^\infty}^{\pi_1^t(\mathcal{U}_s,\eta_s)}$. However, the latter group is finite by lemma 2.7, and this yields a contradiction.

Suppose now that k is a number field. For almost all prime ideals $\mathfrak{q}$ of the ring of integers $\mathcal{O}_k$ of k, the algebraic varieties $\mathcal{A}$ and $\mathcal{C}$ reduce to smooth varieties $\mathcal{A}_\mathfrak{q}$ and $\mathcal{C}_\mathfrak{q}$ over the residue field $\mathbb{F}_\mathfrak{q}$ of $\mathfrak{q}$. Moreover, the choice of $\mathfrak{q}$ can also be made so

that $\phi : \mathcal{A} \to \mathcal{C}$ reduces to a proper flat morphism $\phi_{\mathfrak{q}} : \mathcal{A}_{\mathfrak{q}} \to \mathcal{C}_{\mathfrak{q}}$ also defined over $\mathbb{F}_{\mathfrak{q}}$. Let $K_{\mathfrak{q}} := \mathbb{F}_{\mathfrak{q}}(\mathcal{C}_{\mathfrak{q}})$ be the function field of $\mathcal{C}_{\mathfrak{q}}$ and $K_{\mathfrak{q}}^s$ a separable closure of $K_{\mathfrak{q}}$. The generic fiber of $\phi_{\mathfrak{q}}$ will be a non constant abelian variety $A_{\mathfrak{q}}$ defined over $K_{\mathfrak{q}}$. Let $(\tau_{\mathfrak{q}}, B_{\mathfrak{q}})$ be the $K_{\mathfrak{q}}/\mathbb{F}_{\mathfrak{q}}$-trace of $A_{\mathfrak{q}}$.

By proper base change [Mi80, chapter VI, corollary 2.7]

$$H^1_{\text{ét}}(A \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^s), \mathbb{Z}_p) \cong H^1_{\text{ét}}(A_{\mathfrak{q}} \times_{\mathrm{Spec}(K_{\mathfrak{q}})} \mathrm{Spec}(K^s_{\mathfrak{q}}), \mathbb{Z}_p). \tag{4.3.4}$$

It follows from [Mi85av, theorem 15.1]that

$$\begin{aligned} &H^1_{\text{ét}}(A \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^s), \mathbb{Q}_p/\mathbb{Z}_p) \\ &\cong H^1_{\text{ét}}(A \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^s), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \\ &\cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(A), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \cong \mathrm{Hom}_{\mathbb{Q}_p/\mathbb{Z}_p}(A[p^\infty], \mathbb{Q}_p/\mathbb{Z}_p) \cong A[p^\infty], \end{aligned} \tag{4.3.5}$$

where the latter isomorphism is not canonical, it is just an abstract isomorphism of $\mathbb{Q}_p/\mathbb{Z}_p$-modules. Similarly,

$$H^1_{\text{ét}}(A_{\mathfrak{q}} \times_{\mathrm{Spec}(K_{\mathfrak{q}})} \mathrm{Spec}(K^s_{\mathfrak{q}}), \mathbb{Q}_p/\mathbb{Z}_p) \cong A_{\mathfrak{q}}[p^\infty]. \tag{4.3.6}$$

It now follows from (4.3.4), (4.3.5) and (4.3.6) that $A[p^\infty] \cong A_{\mathfrak{q}}[p^\infty]$. The same argument used above also gives $\tau B[p^\infty] \cong \tau_{\mathfrak{q}} B_{\mathfrak{q}}[p^\infty]$. *A fortiori*,

$$F_{0,p^\infty} = A[p^\infty]/\tau B[p^\infty] \cong A_{\mathfrak{q}}[p^\infty]/\tau_{\mathfrak{q}} B_{\mathfrak{q}}[p^\infty] =: F_{\mathfrak{q},0,p^\infty}. \tag{4.3.7}$$

Let $\mathcal{U}_{\mathfrak{q}}$ be the reduction of $\mathcal{U}$ modulo $\mathfrak{q}$ (it will be equal to the smooth locus of $\phi_{\mathfrak{q}}$ for a generic choice of $\mathfrak{q}$) and $\eta_{\mathfrak{q},s}$ a geometric generic point of $\mathcal{U}_{\mathfrak{q}}$. The reduction $\mathcal{U}_s \to \mathcal{U}_{\mathfrak{q},s}$ modulo $\mathfrak{q}$ induces the specialization homomorphism $\mathrm{sp}_{\mathfrak{q}} : \pi_1^t(\mathcal{U}_s, \eta_s) \cong \pi_1(\mathcal{U}_s, \eta_s) \to \pi_1^t(\mathcal{U}_{\mathfrak{q},s}, \eta_{\mathfrak{q},s})$ at the level of tame fundamental groups. This homomorphism is necessarily surjective by [Gr71, exp. XIII, corollaire 2.12].

Let $\mathcal{G}_{\infty,\mathfrak{q}}$ be the Galois group of a geometric $\mathbb{Z}_p$-extension of the function field $\overline{\mathbb{F}}_{\mathfrak{q}}(\mathcal{C})$. Then we have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{H}_\infty & \longrightarrow & \pi_1(\mathcal{U}_s, \eta_s) & \longrightarrow & \mathcal{G}_\infty & \longrightarrow & 1 \\ & & {\scriptstyle \mathrm{sp}'_{\mathfrak{q},\infty}}\downarrow & & {\scriptstyle \mathrm{sp}_{\mathfrak{q}}}\downarrow & & {\scriptstyle \mathrm{sp}_{\mathfrak{q},\infty}}\downarrow & & \\ 1 & \longrightarrow & \mathcal{H}_{\mathfrak{q},\infty} & \longrightarrow & \pi_1^t(\mathcal{U}_{\mathfrak{q},s}, \eta_{\mathfrak{q},s}) & \longrightarrow & \mathcal{G}_{\mathfrak{q},\infty} & \longrightarrow & 1 \end{array}.$$

Observe that the the group isomorphism $\mathrm{sp}_{\mathfrak{q},\infty}$ is actually obtained via the specialization homomorphism $\mathrm{sp}_{\mathfrak{q}}$. In fact, $\mathrm{sp}_{\mathfrak{q}}$ induces an isomorphism $\mathrm{sp}_{\mathfrak{q}}^{(q')} : \pi_1(\mathcal{U}_s, \eta_s)^{(q')} \xrightarrow{\cong} \pi_1^t(\mathcal{U}_{\mathfrak{q},s}, \eta_{\mathfrak{q},s})^{(q')}$ between the maximal prime to q quotients of both tame fundamental groups (*cf.* [Gr71, exp. X, corollaire 3.9]). Consequently, $\mathrm{sp}_{\mathfrak{q},\infty}$ is obtained from $\mathrm{sp}_{\mathfrak{q}}^{(q')}$ by taking quotients on both sides. In particular, the diagram is commutative. Finally, a simple diagram chasing then shows that the first vertical arrow is also a surjection.

The action of $\mathcal{H}_{\mathfrak{q},\infty}$ on $F_{\mathfrak{q},0,p^\infty}$ and the isomorphism $F_{0,p^\infty} \cong F_{\mathfrak{q},0,p^\infty}$, induce an action of $\mathcal{H}_{\mathfrak{q},\infty}$ on F_{0,p^∞}, thus $\ker(\mathcal{H}_\infty \twoheadrightarrow \mathcal{H}_{\mathfrak{q},\infty})$ acts trivially on F_{0,p^∞}. Therefore, $F_{0,p^\infty}^{\mathcal{H}_\infty} \cong F_{0,p^\infty}^{\mathcal{H}_{\mathfrak{q},\infty}} \cong F_{\mathfrak{q},0,p^\infty}^{\mathcal{H}_{\mathfrak{q},\infty}}$. However, the latter group is finite, by the first part of the proof. *A fortiori*, the first one is also finite.

The same argument using the specialization homomorphism of tame fundamental groups also works if k is a one variable function field over a finite field. In fact, one need only to notice that the surjectivity of the specialization homomorphism of tame fundamental groups holds in general. This follows from [Gr71, exp. XIII, corollaire 2.8] and it is enough to extend the result to the one variable function

field case. Finally, an induction argument on the transcendence degree of k over its prime field κ_0, using the specialization homomorphism, allows us to extend the result to any field k finitely generated over κ_0. $\square$

For each finite group $\mathcal{G}_n$, let $\mathcal{G}^{(n)} := \mathrm{Ker}(\mathcal{G}_\infty \twoheadrightarrow \mathcal{G}_n)$ and k_∞ the minimal algebraic extension of k over which all elements of $\mathcal{G}_\infty$ are defined. As in [El06, proposition 4.1] we have the following result.

Proposition 4.4. *Suppose that k is a finitely generated field over its prime field κ_0 of characteristic either zero or $q > 2d+1$, where $d = \dim(A)$. Assume also that $p > (2d-2d_0)(2g-2+\#(\mathcal{X}_s \setminus \mathcal{C}_s)) + f_{A,\mathcal{C}}$ and $p \neq q$. Then there exists an extension l/k_∞ such that*

(1) $\mathrm{Gal}(k^s/l)$ *acts trivially on* $(A(k^s(\mathcal{C}_n))/\tau_n B_n(k^s)) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p$ *for every* $n \geq 0$.
(2) *l is an abelian pro-p extension of a finite extension of k_∞.*

Proof. The proof follows similarly as in [El06, proposition 4.1], however one has to consider the following point. Applying the Hochschild-Serre spectral sequence to the tower $\mathcal{T}$ one gets for every $n \geq 0$ a group homomorphism $f : S(\mathcal{C}_n, F_{n,p^\infty}) \to S(\mathcal{C}_\infty, F_{\infty,p^\infty})^{\mathcal{G}_n}$ whose kernel is equal to

$$H^1(\mathcal{G}^{(n)}, H^0_{\text{ét}}(\mathcal{C}_{\infty,s}, \mathcal{F}_{0.\infty})) = H^1(\mathcal{G}^{(n)}, F_{0,p^\infty}^{\mathcal{H}_\infty}).$$

But, by proposition 4.3, $F_{0,p^\infty}^{\mathcal{H}_\infty}$ is finite. This replaces the argument of [El06, remark 2.4] in the proof of the proposition. $\square$

As a consequence of proposition 4.4 we obtain the following result.

Theorem 4.5. *(*cf. [El06, theorem 4.4]*) Let $K = k(\mathcal{C})$ be the function field of a smooth geometrically connected curve. Let A/K be a non constant abelian variety of dimension d. Assume that k has characteristic either zero or $q > 2d+1$. Suppose furthermore that $p > (2d-2d_0)(2g-2+\#(\mathcal{X}_s \setminus \mathcal{C}_s)) + f_{A,\mathcal{C}}$ and $p \neq q$. Under the additional hypothesis :*

(†) *for every extension l/k which is an abelian pro-p extension of a finite extension of k_∞, no divisible subgroup of $S(\mathcal{C}, F_{0,p^\infty})$ is fixed by* $\mathrm{Gal}(k^s/l)$*;*

the abelian group $A(k^s(\mathcal{C}_\infty))/\tau_\infty B_\infty(k^s)$ is finitely generated.

Remark 4.6. Once again it is necessary to replace [El06, remark 2.4] in the proof of [El06, theorem 4.4] by proposition 4.3 to get theorem 4.5.

Remark 4.7. In [El06, remark 4.5] it is discussed abstractly condition (†). More precisely, given a cofinitely generated $\mathbb{Z}_p$-module M with an action by $\mathrm{Gal}(k^s/k_\infty)$, condition (†) means that the following property is satisfied. For every extension l of k_∞ which is an abelian pro-p extension of a finite extension of k_∞, no divisible submodule of M is fixed by $\mathrm{Gal}(k^s/l)$. This property is inherited by submodules of M as well as quotients of M by finite submodules. It also respects exact sequences of modules $0 \to M \to M' \to M''$ in the sense that if it holds for M and M'', then it also holds for M'.

5. Jacobian fibrations

5.1. **Generalities.** In [El06, §5] an example was given in which condition (†) is fulfilled for minimal elliptic K3 surfaces. It is natural to ask whether such an example can be produced for higher dimensional abelian fibrations. In this section

we give a necessary condition for the existence of such an example in the context of Jacobian fibrations. However, due to the lack of examples of families of surfaces whose monodromy is "sufficiently large" (*cf.* subsection 5.5) we were not able to produce a concrete example as in the case of elliptic fibrations.

In this section we will assume that $\mathcal{C}$ is a complete smooth geometrically irreducible curve defined over a subfield k of $\mathbb{C}$. For any variety $\mathcal{Y}$ defined over k, denote $\overline{\mathcal{Y}} := \mathcal{Y} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$. If $\mathcal{Z}$ is a variety defined over $K := k(\mathcal{C})$, denote $\overline{\mathcal{Z}} := \mathcal{Z} \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$.

Let $\mathcal{X}$ be a smooth projective geometrically irreducible surface defined over k and $\phi : \mathcal{X} \to \mathcal{C}$ be a proper flat morphism also defined over k. The generic fiber X of ϕ is a smooth projective geometrically irreducible curve defined over K of genus d which we assume to be at least 2. The Jacobian fibration ϕ_J associated to ϕ is a proper flat morphism $\phi_J : \mathcal{J} \to \mathcal{C}$ defined over k from a smooth geometrically irreducible $(d+1)$-fold $\mathcal{J}$ defined over k whose generic fiber is the Jacobian variety $A := \mathrm{Jac}(X)$ of X. It has the property that for every $v \in \mathcal{C}$ for which the fiber $\mathcal{X}_v = \phi^{-1}(v)$ is smooth, then the fiber $\phi_J^{-1}(v)$ coincides with the Jacobian variety $\mathrm{Jac}(\mathcal{X}_v)$ of $\mathcal{X}_v$. Let (τ, B) be the K/k-trace of A.

As before, given a prime number p, let $F_{p^\infty} := A[p^\infty]/\tau B[p^\infty]$. Let $\overline{\eta}$ be the geometric generic point of $\mathcal{C}$, $j : \overline{\eta} \hookrightarrow \overline{\mathcal{C}}$ the inclusion map and $\mathcal{F}_{p^\infty} := j_*(F_{p^\infty})$. Let $\tilde{\mathcal{F}}_{p^\infty} := R^1\phi_*(\mathbb{Q}_p/\mathbb{Z}_p)$.

Suppose that (τ, B) is trivial. It follows from [Mi85jv, corollary 9.6] after tensoring with $\mathbb{Q}_p/\mathbb{Z}_p$ that

$$H^1_{\text{ét}}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p) \cong H^1_{\text{ét}}(\overline{A}, \mathbb{Q}_p/\mathbb{Z}_p).$$

It then follows from [Mi85av, theorem 15.1]that

$$\begin{aligned} H^1_{\text{ét}}(\overline{A}, \mathbb{Q}_p/\mathbb{Z}_p) \cong H^1_{\text{ét}}(\overline{A}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p &\cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(A), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \\ &\cong \mathrm{Hom}_{\mathbb{Q}_p/\mathbb{Z}_p}(A[p^\infty], \mathbb{Q}_p/\mathbb{Z}_p) \cong A[p^\infty], \end{aligned}$$

where the latter isomorphism is not canonical, just as abstract $\mathbb{Q}_p/\mathbb{Z}_p$-modules. As a consequence,

$$(5.1.1) \quad j^*(\tilde{\mathcal{F}}_{p^\infty}) = (\tilde{\mathcal{F}}_{p^\infty})_{\overline{\eta}} = H^1_{\text{ét}}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p) \cong H^1_{\text{ét}}(\overline{A}, \mathbb{Q}_p/\mathbb{Z}_p) \cong A[p^\infty] = F_{p^\infty},$$

whence $j_*(j^*(\tilde{\mathcal{F}}_{p^\infty})) \cong \mathcal{F}_{p^\infty}$. Therefore, we have a surjective map $H^1_{\text{ét}}(\overline{\mathcal{C}}, \tilde{\mathcal{F}}_{p^\infty}) \to H^1_{\text{ét}}(\overline{\mathcal{C}}, \mathcal{F}_{p^\infty}) = S(\mathcal{C}, F_{p^\infty})$, since the kernel of $\tilde{\mathcal{F}}_{p^\infty} \to \mathcal{F}_{p^\infty}$ has zero dimensional support.

The Leray spectral sequence $H^i_{\text{ét}}(\overline{\mathcal{C}}, R^j\phi_*(\mathbb{Q}_p/\mathbb{Z}_p)) \implies H^{i+j}_{\text{ét}}(\overline{\mathcal{X}}, \mathbb{Q}_p/\mathbb{Z}_p)$ yields the exact sequence of cohomology groups

$$\begin{aligned} (5.1.2) \quad 0 \to H^1_{\text{ét}}(\overline{\mathcal{C}}, \mathbb{Q}_p/\mathbb{Z}_p) \to H^1_{\text{ét}}(\overline{\mathcal{X}}, \mathbb{Q}_p/\mathbb{Z}_p) \to H^0_{\text{ét}}(\overline{\mathcal{C}}, \tilde{\mathcal{F}}_{p^\infty}) \to \\ H^2_{\text{ét}}(\overline{\mathcal{C}}, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{d_2} H^2_{\text{ét}}(\overline{\mathcal{X}}, \mathbb{Q}_p/\mathbb{Z}_p). \end{aligned}$$

Let F be a smooth fiber of ϕ. Then the image of d_2 in (5.1.2) is generated by the class $[F]$ of F in $H^2_{\text{ét}}(\overline{\mathcal{X}}, \mathbb{Q}_p/\mathbb{Z}_p)$. Let M be the quotient of the latter group by the subgroup generated by $[F]$. Now the previous spectral sequence at degree 2 yields

$$(5.1.3) \qquad 0 \to H^1_{\text{ét}}(\overline{\mathcal{C}}, \tilde{\mathcal{F}}_{p^\infty}) \xrightarrow{d_1} M \to H^0_{\text{ét}}(\overline{\mathcal{C}}, R^2\phi_*(\mathbb{Q}_p/\mathbb{Z}_p)) \to H^2_{\text{ét}}(\overline{\mathcal{C}}, \tilde{\mathcal{F}}_{p^\infty}).$$

Observe that $H^2_{\text{ét}}(\overline{\mathcal{C}}, \tilde{\mathcal{F}}_{p^\infty})$ is finite, since it is dual to

$$H^0_{\text{ét}}(\overline{\mathcal{C}}, \tilde{\mathcal{F}}_{p^\infty}) \cong H^1_{\text{ét}}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p)^{\pi_1^t(\overline{\mathcal{U}}, \overline{\eta})} \cong A[p^\infty]^{\pi_1^t(\overline{\mathcal{U}}, \overline{\eta})},$$

where the latter isomorphism follows from (5.1.1) and the finiteness of the latter space is a consequence of lemma 2.7. Hence,

$$(5.1.4)\quad \mathrm{corank}_{\mathbb{Z}_p}(H^1_{\text{ét}}(\overline{\mathcal{C}},\tilde{\mathcal{F}}_{p^\infty})) = \mathrm{corank}_{\mathbb{Z}_p}(M) - \mathrm{corank}_{\mathbb{Z}_p}(H^0_{\text{ét}}(\overline{\mathcal{C}}, R^2\phi_*(\mathbb{Q}_p/\mathbb{Z}_p))).$$

Observe that the generic stalk of $R^2\phi_*(\mathbb{Q}_p/\mathbb{Z}_p)$ has corank 1. For each $v \in \overline{\mathcal{C}}$ let m_v be the number of irreducible components of the fiber $\mathcal{X}_v := \phi^{-1}(v)$. Then the corank of $R^2\phi_*(\mathbb{Q}_p/\mathbb{Z}_p)$ at v equals $m_v - 1$. Therefore

$$(5.1.5)\qquad \mathrm{corank}_{\mathbb{Z}_p}(H^0_{\text{ét}}(\overline{\mathcal{C}}, R^2\phi_*(\mathbb{Q}_p/\mathbb{Z}_p))) = 1 + \sum_{v\in\overline{\mathcal{C}}}(m_v - 1).$$

Let $\mathrm{Pic}(\mathcal{X})$ be the Picard group of $\overline{k}$-divisor classes of $\mathcal{X}$. The composition

$$(5.1.6)\qquad M \to H^0_{\text{ét}}(\overline{\mathcal{C}}, R^2\phi_*(\mathbb{Q}_p/\mathbb{Z}_p)) \to H^0_{\text{ét}}(\overline{\mathcal{C}}, j_*(j^*(R^2\phi_*(\mathbb{Q}_p/\mathbb{Z}_p)))) \cong \mathbb{Q}_p/\mathbb{Z}_p$$

is the degree map. It sends the image in $H^2_{\text{ét}}(\overline{\mathcal{X}},\mathbb{Q}_p/\mathbb{Z}_p)$ of a class $[D]$ of a divisor in $\mathrm{Pic}(\mathcal{X})$ to its intersection number $(D\cdot X)$ with the generic fiber. Let $G(\mathcal{X})$ be the quotient of the subset of elements of degree zero in $H^2_{\text{ét}}(\overline{\mathcal{X}},\mathbb{Q}_p/\mathbb{Z}_p)$ by the module generated by the class of $[F]$. Thus, $H^1_{\text{ét}}(\overline{\mathcal{C}},\tilde{\mathcal{F}}_{p^\infty})$ is a submodule of $G(\mathcal{X})$.

Let $(D_0\cdot X)$ be a generator of the ideal $\{(D\cdot X)\,|\, D\in \mathrm{Div}(\mathcal{X})\}$, where $(\ \cdot\ )$ denotes the intersection pairing on $\mathrm{Pic}(\mathcal{X})$. Let $\imath : X \to \mathcal{X}$ be the inclusion map and $\imath^* : \mathrm{Div}(\mathcal{X}) \to \mathrm{Div}(X)$ the pull-back map obtained by restricting the divisors to the generic fiber X. Define $\psi : \mathrm{Pic}(\mathcal{X}) \to A$ by

$$\psi([D]) := \imath^*\left(D - \frac{(D\cdot X)}{(D_0\cdot X)}D_0\right).$$

It follows from [HiPa05, lemme 3.7] that ψ is surjective and its kernel $\mathcal{S}$ is generated by the classes $[D_0]$, $[F]$ and the classes of all irreducible components of the singular fibers of ϕ (except one). *A fortiori*,

$$(5.1.7)\qquad \mathrm{rank}_{\mathbb{Z}}(\mathcal{S}) = 2 + \sum_{v\in\overline{\mathcal{C}}}(m_v - 1).$$

Let $\mathrm{NS}(\mathcal{X})$ be the Néron-Severi group of the surface $\mathcal{X}$. We also extended the Shioda-Tate formula (*cf.* [HiPa05, proposition 3.8]) to fibrations not necessarily having a section and without any hypothesis on the K/k-trace (τ, B) of A being trivial. As a consequence of this result we obtained

$$(5.1.8)\qquad \mathrm{rank}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{X})) = \mathrm{rank}_{\mathbb{Z}}\left(\frac{A(\overline{k}(\mathcal{C}))}{\tau B(\overline{k})}\right) + \mathrm{rank}_{\mathbb{Z}}(\mathcal{S}).$$

The cohomology group $H^2_{\text{ét}}(\overline{\mathcal{X}},\mathbb{Z}_p)$ comes equipped with a quadratic form $q_{\mathcal{X}}$ through the pairing defined by the cup product. Note also that for the images of the elements of $\mathrm{Pic}(\mathcal{X})$ in $H^2_{\text{ét}}(\overline{\mathcal{X}},\mathbb{Z}_p)$, the intersection pairing in $\mathrm{Pic}(\mathcal{X})$ is compatible with the cup product in $H^2_{\text{ét}}(\overline{\mathcal{X}},\mathbb{Z}_p)$. Moreover, $\mathrm{corank}_{\mathbb{Z}_p}(H^2_{\text{ét}}(\overline{\mathcal{X}},\mathbb{Z}_p))$ equals the second Betti number $b_2(\mathcal{X})$ of $\mathcal{X}$. Let Γ'_p be the subgroup of automorphisms of $H^2_{\text{ét}}(\overline{\mathcal{X}},\mathbb{Z}_p)$ which preserves $q_{\mathcal{X}}$ and stabilizes $[F]$ and $[D_0]$. Let Γ_p be a finite index subgroup of Γ'_p.

5.2. **A sufficient condition.**

Theorem 5.3. *(*cf. [El06, theorem 5.1]*) Let $\mathcal{X}$, resp. $\mathcal{C}$, be a smooth projective irreducible surface, resp. curve, defined over a number field k. Let $\phi : \mathcal{X} \to \mathcal{C}$ be a proper flat morphism also defined over k. Let $d \geq 2$ be the genus of the generic fiber X of ϕ. Let p be a prime number, $n \geq 0$ an integer, h a function in $K := k(\mathcal{C})$, $K_n := K(h^{1/p^n})$, $\mathcal{K} := \overline{k}(\mathcal{C})$ and $\overline{K}_n := \mathcal{K}(h^{1/p^n})$. Let $\phi_J : \mathcal{J} \to \mathcal{C}$ be the Jacobian fibration associated to ϕ and A the generic fiber of ϕ_J. Suppose the following*

(a) *The image of* $\mathrm{Gal}(\overline{k}/k)$ *in* $\mathrm{Aut}(H^2_{\acute{e}t}(\overline{\mathcal{X}}, \mathbb{Z}_p))$ *contains* Γ_p.
(b) *d is either 2, 6 or odd and for every $v \in \overline{\mathcal{C}}$ which is not a zero nor a pole of h, the fiber $\mathcal{X}_v$ of ϕ is smooth and* $\mathrm{End}_{\overline{\kappa}_v}(\mathrm{Jac}(\mathcal{X}_v)) = \mathbb{Z}$, *where $\overline{\kappa}_v$ denotes an algebraic closure of the residue field κ_v of v and* $\mathrm{Jac}(\mathcal{X}_v)$ *is the Jacobian variety of $\mathcal{X}_v$.*
(c) $p > b_2(\mathcal{X}) - 2 + 2d \cdot \deg(h)$.

Then the rank of the Mordell-Weil *group $A(\overline{K}_n)$ is uniformly bounded as $n \to \infty$.*

Proof. We start by observing that condition (a) implies that for every $n \geq 1$, the K_n/k-trace of A is trivial. Let $k_\infty := k(\zeta_{p^\infty})$ be the field obtained from k by adjoining all p-th power roots of unity. By the previous condition, the image of $\mathrm{Gal}(\overline{k}/k_\infty)$ in $\mathrm{Aut}(H^2_{\text{ét}}(\overline{\mathcal{X}}), \mathbb{Z}_p)$ still contains a finite index subgroup of Γ_p, since the determinant map sends Γ_p to a finite group. Next, denote by H the $\mathbb{Z}_p$-module generated by the images of the classes of F and D_0. Then $\mathrm{Gal}(\overline{k}/k_\infty)$ acts irreducibly on $(H^2_{\text{ét}}(\overline{\mathcal{X}}, \mathbb{Z}_p)/H) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. It follows from the second exact sequence in p.29 of [Ra68] that we have a surjective map $\psi : H^2_{\text{ét}}(\overline{\mathcal{X}}, \mathbb{Z}_p) \twoheadrightarrow H^2_{\text{ét}}(\overline{\mathcal{X}}, \mathbb{G}_m)\{p^\infty\}$, where the latter denotes the p-primary subgroup of $H^2_{\text{ét}}(\overline{\mathcal{X}}, \mathbb{G}_m)$. As a consequence $\mathrm{Gal}(\overline{k}/k_\infty)$ also acts irreducibly on $(H^2_{\text{ét}}(\overline{\mathcal{X}}, \mathbb{G}_m)\{p^\infty\}/\psi(H)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Recall the inclusion map $j : \overline{\eta} \hookrightarrow \mathcal{C}$. It follows from the last map of p.28 of [Ra68] that there exists an isomorphism $\vartheta : H^1_{\text{ét}}(\overline{\mathcal{C}}, j_*A)\{p^\infty\} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\cong} H^2_{\text{ét}}(\overline{\mathcal{X}}, \mathbb{G}_m)\{p^\infty\} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where in the left hand side we are considering A as an étale sheaf on $\overline{\eta}$. *A fortiori*, $\mathrm{Gal}(\overline{k}/k_\infty)$ also acts irreducibly on $(H^1_{\text{ét}}(\overline{\mathcal{C}}, j_*A)\{p^\infty\} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)/\vartheta^{-1}(\psi(H) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$. However, if the K/k-trace (τ, B) of A is not zero, then this cannot happen, since the latter $\mathbb{Q}_p[\mathrm{Gal}(\overline{k}/k_\infty)]$-module admits $(H^1_{\text{ét}}(\overline{\mathcal{C}}, j_*(\tau(B)))\{p^\infty\} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)/((H^1_{\text{ét}}(\overline{\mathcal{C}}, j_*(\tau(B)))\{p^\infty\} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \cap \vartheta^{-1}(\psi(H) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ as a $\mathbb{Q}_p[\mathrm{Gal}(\overline{k}/k_\infty)]$-submodule. Hence, $B = 0$. The same argument applies the K_n/k-trace of A for every $n \geq 1$.

As a consequence of (5.1.4) and (5.1.5) we have

$$\operatorname{corank}_{\mathbb{Z}_p}(H^1_{\text{ét}}(\overline{\mathcal{C}}, \tilde{\mathcal{F}}_{p^\infty})) = b_2(\mathcal{X}) - 2 - \sum_{v \in \overline{\mathcal{C}}} (m_v - 1). \tag{5.3.1}$$

The second thing we need to do is to show that this corank equals $2d(2g-2) + f_A$. Indeed, it follows from the discussion on [Ra68, §3] and the formula [Ra68, théorème 3, (ii)] that

$$b_2(\mathcal{X}) - \operatorname{rank}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{X})) = -\operatorname{rank}(A(\overline{k}(\mathcal{C}))) + 2d(2g-2) + f_A. \tag{5.3.2}$$

Hence, by (5.1.8), (5.1.7), (5.3.2) and 5.3.1, we have

$$\operatorname{corank}_{\mathbb{Z}_p}(H^1_{\text{ét}}(\overline{\mathcal{C}}, \tilde{\mathcal{F}}_{p^\infty})) = 2d(2g-2) + f_A. \tag{5.3.3}$$

(In fact the previous calculations would also hold if the $B \neq 0$, however we would need to add $4\dim(B)$ to the right hand side of (5.3.3).)

The rest of the proof is very similar to that of [El06, theorem 5.1], we restrict ourselves to just pointing out the differences between them. As in *loc. cit.*, $G(\mathcal{X})$ satisfies property (†), hence so does its submodule $H^1_{\text{ét}}(\overline{\mathcal{C}}, \tilde{\mathcal{F}}_{p^\infty})$ (*cf.* remark 4.7). Since the corank of this module equals $2d(2g-2)+f_A$ which is the corank of $S(\mathcal{C}, F_{p^\infty})$, we conclude that the surjection $H^1_{\text{ét}}(\overline{\mathcal{C}}, \tilde{\mathcal{F}}_{p^\infty}) \twoheadrightarrow H^1_{\text{ét}}(\overline{\mathcal{C}}, \mathcal{F}_{p^\infty})$ has finite kernel. *A fortiori* (*cf.* remark 4.7), $H^1_{\text{ét}}(\overline{\mathcal{C}}, \mathcal{F}_{p^\infty})$ also satisfies condition (†).

Let $\overline{\mathcal{Z}} \subset \overline{\mathcal{C}}$ be the scheme of zeroes and poles of h in $\overline{\mathcal{C}}$. Let $\overline{\mathcal{V}} := \overline{\mathcal{C}} \setminus \overline{\mathcal{Z}}$. Whence we have an exact sequence (*cf.* [Mi80, chapter III, proposition 1.25])

$$(5.3.4)\quad 0 \to H^1_{\text{ét}}(\overline{\mathcal{C}}, \mathcal{F}_{p^\infty}) \to H^1_{\text{ét}}(\overline{\mathcal{V}}, \mathcal{F}_{p^\infty}) \to H^0_{\text{ét}}(\overline{\mathcal{Z}}, \mathcal{F}_{p^\infty}(-1)_{|\overline{\mathcal{Z}}}) = \bigoplus_{v\in\overline{\mathcal{Z}}} H^1_{\text{ét}}(\mathcal{X}_v, (\mathbb{Q}_p/\mathbb{Z}_p)(-1)),$$

where in the last equality of (5.3.4) we used Poincaré's duality for étale cohomology.

It follows from hypothesis (a) that the action of $\mathrm{Gal}(\overline{k}/k_\infty)$ on $G(\mathcal{X})$ is irreducible. Hence all fibers of ϕ are irreducible. It follows from [Mi85jv, proposition 9.6] that there exists an isomorphism (after having tensored both sides by $(\mathbb{Q}_p/\mathbb{Z}_p)(-1)$)

$$H^1_{\text{ét}}(\overline{\mathcal{X}}_v, (\mathbb{Q}_p/\mathbb{Z}_p)(-1)) \cong H^1_{\text{ét}}(J_{\overline{\mathcal{X}}_v}, (\mathbb{Q}_p/\mathbb{Z}_p)(-1)).$$

But by hypothesis (b), the MUMFORD-TATE conjecture is true for $\mathrm{Jac}(\overline{\mathcal{X}}_v)$ (*cf.* remark 5.4). Consequently, $H^0_{\text{ét}}(\overline{\mathcal{Z}}, \mathcal{F}_{p^\infty}(-1)_{|\overline{\mathcal{Z}}})$ also satisfies property (†). In particular, by remark 4.7, $H^1_{\text{ét}}(\overline{\mathcal{V}}, \mathcal{F}_{p^\infty}) = S(\mathcal{V}, F_{p^\infty})$ also satisfies property (†). Moreover by (5.3.4) its $\mathbb{Z}_p$-corank is at most equal to $b_2(\mathcal{X}) - 2 + 2d\cdot\deg(h) < p$. The choice of $\mathcal{V}$ implies that the map $h \mapsto h^{1/p^n}$ at the level of functions gives an étale Galois cover $\mathcal{V}_n \to \mathcal{V}$ at the level of curves and we are exactly in the set-up of theorem 4.5, where $\mathcal{V}$ plays the role of the affine curve in that statement. The theorem is now a consequence of the latter result. □

Remark 5.4. We introduced condition (b) in the hypotheses of the theorem to assure the truth of the MUMFORD-TATE conjecture for the fibers $\mathcal{J}_v$ of ϕ_J for $v \in \overline{\mathcal{C}}$ outside the support of the divisor of h. For a discussion on the MUMFORD-TATE conjecture see [Pi98]. For us here the only important thing is the following consequence. Suppose that Y is an abelian variety of dimension d defined over a number field k. Assume that k is sufficiently large so that Y is principally polarized. For each integer $n \geq 1$, let $Y[n]$ be the subgroup of n-torsion points of Y. SERRE proved in [SeCF85, théorème 3] that if $\mathrm{End}_{\overline{k}}(Y) = \mathbb{Z}$ and its dimension is either 2, 6 or odd, then the image of the GALOIS representation $\rho_n : \mathrm{Gal}(\overline{k}/k) \to \mathrm{Aut}(Y[n]) \cong \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z})$ has index at most $I(Y,k)$ (independent of n) for every $n \geq 1$ (*cf.* [Pa05, theorem 6.3]).

5.5. **Towards families of surfaces.** The hardest condition on theorem 5.3 is (a). The idea to produce an example where it might be satisfied is the following. Suppose that there exists a proper flat family of surfaces $\pi : \mathfrak{X} \to \mathfrak{S}$ parametrized by a smooth projective variety $\mathfrak{S}$. Assume that $\mathfrak{X}$, $\mathfrak{S}$ and π are defined over k. Suppose furthermore that for every $s \in \mathfrak{S}$ the fiber $\mathfrak{X}_s$ of π at s comes equipped with a fibration on curves $\phi_s : \mathfrak{X}_s \to \mathcal{C}_s$ to a smooth projective geometrically connected

curve $\mathcal{C}_s$. Let $L = k(\mathfrak{S})$ be the function field of $\mathfrak{S}$, $\overline{L}$ an algebraic closure of L, X/L the generic fiber of π (which is also assumed to be projective and geometrically connected), $\overline{X} := X \times_{\mathrm{Spec}(L)} \mathrm{Spec}(L)$ and η a geometric generic point of $\mathfrak{S}$. Denote by $\pi_1^{\mathrm{geom}} := \pi_1(\overline{X}, \eta)$ the geometric algebraic fundamental group of X with respect to η. Consider the monodromy representation $\rho : \pi_1^{\mathrm{geom}} \to \mathrm{Aut}(H^2_{\text{ét}}(\overline{X}, \mathbb{Z}_p))$ and let G^{geom} be its geometric monodromy group, *i.e.*, the ZARISKI closure of the image of ρ in the algebraic group $\mathrm{GL}_{N,\mathbb{Z}_p}$, where $N = \mathrm{rank}_{\mathbb{Z}_p}(H^2_{\text{ét}}(\overline{X}, \mathbb{Z}_p))$. If we can prove that G^{geom} is the orthogonal algebraic group $O_{N,\mathbb{Z}_p}$, then a similar argument to that in [El06, §5] (supposing that k is a number field and using the HILBERT's irreducibility theorem) allows one to obtain for u in an open subset U of $\mathfrak{S}$ a representation $\rho_u : \mathrm{Gal}(\overline{k}/k) \to \mathrm{Aut}(H^2_{\text{ét}}(\overline{\mathfrak{X}}_u, \mathbb{Z}_p))$ induced by ρ with the following property (here $\mathfrak{X}_u$ denotes the element in the family π corresponding to the fiber at u). The image of ρ_u must contain a subgroup of finite index of the group of automorphisms of $H^2_{\text{ét}}(\overline{\mathfrak{X}}_u, \mathbb{Z}_p)$ which preserves the quadratic form defined by the cup product and stabilizes a fiber F of ϕ_u and horizontal divisor D_0 in $\mathfrak{X}_u$ (with notation as in the previous subsection). This would give an infinite number of surfaces satisfying condition (a). However, examples of families of surfaces whose geometric monodromy group is the orthogonal group seem still to be lacking.

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Universidade Federal do Rio de Janeiro, Instituto de Matemática, Rua Guaiaquil 83, Cachambi, 20785-050 Rio de Janeiro, RJ, Brasil

E-mail address: `amilcar@acd.ufrj.br`